

# Elimination Theory in Codimension Two

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## Abstract

New formulas are given for Chow forms, discriminants and resultants arising from (not necessarily normal) toric varieties of codimension 2. The Newton polygon of the discriminant is determined exactly.

## 1 Introduction

Sparse elimination theory concerns the study of Chow forms and discriminants associated with toric varieties, that is, subvarieties of projective space which are parametrized by monomials [4], [9]. This theory has its origin in the work of Gel'fand, Kapranov and Zelevinsky on multivariate hypergeometric functions [3]. The singularities of these functions occur on the projectively dual hypersurfaces to the torus orbit closures on the given toric variety  $X$ . The singular locus of the hypergeometric system is described by the full discriminant of  $X$ , which is a natural specialization of the Chow form.

Classical hypergeometric functions in one variable arise when  $X$  is a toric hypersurface, defined by one homogeneous binomial equation  $x_1^{b_1} \cdots x_r^{b_r} = x_{r+1}^{b_{r+1}} \cdots x_n^{b_n}$ . The Chow form of this hypersurface  $X$  is just its defining polynomial. The discriminant of  $X$  equals, up to an integer factor, [4, §9.1],

$$D_X = b_{r+1}^{b_{r+1}} \cdots b_n^{b_n} \cdot x_1^{b_1} \cdots x_r^{b_r} - (-1)^{\deg(X)} b_1^{b_1} \cdots b_r^{b_r} \cdot x_{r+1}^{b_{r+1}} \cdots x_n^{b_n}, \quad (1.1)$$

and the full discriminant equals  $D_X$  times  $\prod_{i=1}^n x_i^{\deg(X)-b_i}$ . It is the purpose of this article to generalize these formulas to toric varieties of codimension 2. Our motivations for this study include hypergeometric functions [1], Horn systems in two variables [8], and their applications to theoretical physics [5].

We introduce our objects of study by means of an example. Let  $X$  be the toric 6-fold in projective 8-space given parametrically by the cubic monomials

$$(a : b : \cdots : i) = (u_1 u_0^2 : u_2 u_0^2 : u_3 u_0^2 : u_1 x^2 : u_2 y^2 : u_3 z^2 : u_4 yz : u_4 xz : u_4 xy).$$

The prime ideal of the toric variety  $X$  is generated by the  $2 \times 2$ -minors of

$$\begin{pmatrix} a & b & c \\ dg^2 & eh^2 & fi^2 \end{pmatrix} \quad (1.2)$$

Thus  $X$  is arithmetically Cohen-Macaulay and has degree 13. The Chow form of  $X$  is gotten by eliminating the variable  $t$  from the  $2 \times 2$ -minors of

$$\begin{pmatrix} a_0 + ta_1 & b_0 + tb_1 & c_0 + tc_1 \\ (d_0 + td_1)(g_0 + tg_1)^2 & (e_0 + te_1)(h_0 + th_1)^2 & (f_0 + tf_1)(i_0 + ti_1)^2 \end{pmatrix} \quad (1.3)$$

The Chow form is an irreducible polynomial of degree 26 in the 18 variables  $a_0, a_1, b_0, b_1, \dots, i_0, i_1$  having exactly 57,726 terms. It equals the determinant

$$\begin{pmatrix} 123 & 124 & 125 & 126 \\ 134 & 135 + 234 & 136 + 235 & 236 \\ 135 & 136 + 145 + 235 & 146 + 236 + 245 & 246 \\ 136 & 146 + 236 & 156 + 246 & 256 \end{pmatrix} \quad (1.4)$$

where  $ijk$  is the  $3 \times 3$ -minor with row indices  $i, j$  and  $k$  of the  $6 \times 3$ -matrix

$$\begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ d_0 g_0^2 & e_0 h_0^2 & f_0 i_0^2 \\ d_1 g_0^2 + 2d_0 g_0 g_1 & e_1 h_0^2 + 2e_0 h_0 h_1 & f_1 i_0^2 + 2f_0 i_0 i_1 \\ d_0 g_1^2 + 2d_1 g_0 g_1 & e_0 h_1^2 + 2e_1 h_0 h_1 & f_0 i_1^2 + 2f_1 i_0 i_1 \\ d_1 g_1^2 & e_1 h_1^2 & f_1 i_1^2 \end{pmatrix} \quad (1.5)$$

Note that the Chow form can also be written as a polynomial of degree 13 in the brackets  $[ab] = a_0 b_1 - a_1 b_0$ ,  $[ac] = a_0 c_1 - a_1 c_0$ ,  $\dots$ ,  $[hi] = h_0 i_1 - h_1 i_0$ . We obtain the full discriminant of  $X$  from the Chow form by substituting

$$\begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \\ \vdots & \vdots \\ i_0 & i_1 \end{pmatrix} \mapsto \text{diag}(a, b, c, d, e, f, g, h, i) \cdot B, \quad (1.6)$$

where  $B$  is the  $9 \times 2$ -matrix with row vectors  $(1, 0), (0, 1), (-1, -1), (-1, 0), (0, -1), (1, 1), (-2, 0), (0, -2), (2, 2)$ . The result of this substitution is the dual full discriminant  $\tilde{E}_X$ . It has exactly twelve terms and factors as follows:

$$\tilde{E}_X = 2^{14} \cdot (aeh^2 - bdg^2) \cdot (afi^2 - cdg^2) \cdot (bfi^2 - ceh^2) \cdot \tilde{D}_X, \quad (1.7)$$

where the last factor  $\tilde{D}_X$  is the irreducible polynomial

$$\begin{aligned} & a^2e^2f^2h^4i^4 + b^2d^2f^2g^4i^4 + c^2d^2e^2g^4h^4 \\ & - 2abdef^2g^2h^2i^4 - 2acdf e^2g^2h^4i^2 - 2bcef d^2g^4h^2i^2 \end{aligned}$$

Replacing each variable in  $\tilde{D}_X$  by its reciprocal, that is,  $a \mapsto 1/a, b \mapsto 1/b, \dots$  and clearing denominators, we get the discriminant  $D_X$ , an irreducible polynomial of degree 10 which defines the hypersurface projectively dual to  $X$ .

In this paper we establish exact formulas for the Chow form (Theorems 2.1 and 2.7), the full discriminant (Proposition 3.2), and the discriminant (Theorem 4.2) associated with an arbitrary toric variety  $X$  of codimension 2 in a projective space. A combinatorial construction is given for the secondary polygon (Theorem 3.4) and the Newton polygon of the discriminant (Theorem 4.3). This construction shows that the dual variety  $X^\vee$  is a hypersurface if and only if the secondary polygon is not centrally symmetric (Corollary 4.5). In Section 5 we study mixed resultants, that is, we apply our theory to codimension 2 toric varieties which arise from the Cayley trick [4, §3.2.D]

The toric 6-fold  $X$  in our example does arise from the Cayley trick. This can be seen from the defining parametrization  $(u_1u_0^2 : \dots : u_4xy)$ . Hence the discriminant  $D_X$  is actually a resultant. Indeed, if we eliminate  $x, y, z$  from

$$a + d \cdot x^2 = b + e \cdot y^2 = c + f \cdot z^2 = g \cdot yz + h \cdot xz + i \cdot xy = 0 \quad (1.8)$$

then the result is precisely the six-term discriminant  $D_X$  described above.

## 2 The Chow form

Let  $B = (b_{i\ell})$  be an  $n \times 2$ -integer matrix of rank 2 with both column sums equal to zero. The *lattice ideal*  $I_B$  is the ideal in  $k[x_1, \dots, x_n]$ ,  $k$  a field of characteristic zero, generated by the binomials  $x^{u_+} - x^{u_-}$  where  $u = u_+ - u_-$  runs over the two-dimensional lattice  $L_B \subset \mathbb{Z}^n$  spanned by the columns of  $B$ . The minimal generators and the higher syzygies of  $I_B$  are described explicitly

in [7]. The ideal  $I_B$  is homogeneous with respect to the usual  $\mathbb{Z}$ -grading and hence defines a subscheme  $X_B$  of projective space  $\mathbf{P}^{n-1}$ . The lattice ideal  $I_B$  is prime if and only if  $\mathbb{Z}^n/L_B$  is a free abelian group, or equivalently, if and only if the rows of  $B$  generate the two-dimensional lattice  $\mathbb{Z}^2$ .

In this section we compute the Chow form and the Chow polygon of the projective scheme  $X_B$ . The *degree* of  $X_B$ , denoted  $d_B = \text{degree}(X_B)$ , is the number of intersection points with a generic 2-plane in  $\mathbf{P}^{n-1}$ . Let  $Y = (y_{i\ell})$  be an  $n \times 2$ -matrix of indeterminates. It represents a generic parametric line  $(y_{11} + ty_{12}, \dots, y_{n1} + ty_{n2})$  in  $\mathbf{P}^{n-1}$ . Following [4, §3.2.B], the *Chow form*  $\tilde{C}_B$  of the homogeneous lattice ideal  $I_B$  is the unique (up to sign) irreducible homogeneous polynomial in  $\mathbb{Z}[y_{i\ell}]$  which vanishes if and only if the corresponding line in  $\mathbf{P}^{n-1}$  meets  $X_B$ . The degree of  $\tilde{C}_B$  equals  $2 \cdot d_B$ .

Classical invariant theory (cf. [4, Proposition 3.1.6]) tells us that the Chow form  $\tilde{C}_B$  can be written (non-uniquely) as a polynomial of degree  $d_B$  in the (*dual*) *Plücker coordinates* of a generic line, which we write as brackets

$$[i\ j] \quad := \quad y_{i1}y_{j2} - y_{i2}y_{j1} \quad \text{for } 1 \leq i < j \leq n.$$

We further introduce a non-negative integer  $\nu_{ij}$  for any  $1 \leq i < j \leq n$  as follows: if the  $i$ -th row vector and the  $j$ -th row vector of  $B = (b_{i\ell})$  have the same sign in one of the two coordinates then set  $\nu_{ij} = 0$ ; otherwise we set

$$\nu_{ij} \quad := \quad \min \{ |b_{i1}b_{j2}|, |b_{i2}b_{j1}| \}. \quad (2.1)$$

Thus,  $\nu_{ij} = 0$  unless  $b_i$  and  $b_j$  lie in the interior of opposite quadrants. Let

$$H_\ell(t) = \prod_{i:b_{i\ell}>0} (y_{i1} + y_{i2}t)^{b_{i\ell}} - \prod_{i:b_{i\ell}<0} (y_{i1} + y_{i2}t)^{-b_{i\ell}}, \quad \ell = 1, 2. \quad (2.2)$$

We regard  $H_1$  and  $H_2$  as polynomials in a single variable  $t$  with coefficients in  $\mathbb{Z}[y_{i\ell}, i = 1, \dots, n, \ell = 1, 2]$ . Let  $\beta_\ell$  denote the sum of the positive entries in the  $\ell$ -th column of  $B$ , for  $\ell = 1, 2$ . Clearly,  $\text{degree}(h_\ell) = \beta_\ell, \ell = 1, 2$ .

**Theorem 2.1.** *The Chow form of the codimension 2 lattice ideal  $I_B$  equals*

$$\tilde{C}_B = \frac{\text{Res}_t(H_1, H_2)}{\prod_{1 \leq r < s \leq n} [r\ s]^{\nu_{rs}}},$$

where  $\text{Res}_t$  denotes the Sylvester resultant of two univariate polynomials.

*Proof.* The binomials  $\prod_{b_{ij}>0} x_i^{b_{ij}} - \prod_{b_{ij}<0} x_i^{-b_{ij}}$ ,  $j = 1, 2$ , defined by the two columns of  $B$  determine a complete intersection  $Y_B$  of degree  $\beta_1\beta_2$  in  $\mathbf{P}^{n-1}$  which coincides with  $X_B$  over  $(k^*)^{n-1}$ . The irreducible decomposition of  $Y_B$  consists of the components of  $X_B$  – of which there is only one if  $\mathbb{Z}^n/L_B$  is free abelian – together with subschemes supported on coordinate flats  $x_r = x_s = 0$ , whose Chow forms are the bracket monomials  $[r\ s]$ . The theorem will be proved if we show that the cycle  $\{x_r = x_s = 0\}$  occurs with multiplicity  $\nu_{rs}$  in the complete intersection.

Suppose first that  $\nu_{rs} = 0$ . We may assume that  $b_{r1}, b_{s1} \geq 0$ . Then,  $\{x_r = x_s = 0\}$  is not contained in  $Y_B$ , and thus occurs with multiplicity 0. Suppose now that  $\nu_{rs} > 0$ . We may assume that  $b_{r1}, b_{r2} > 0$  and  $b_{s1}, b_{s2} < 0$ . Then,  $\{x_r = x_s = 0\}$  is contained in  $Y_B$ , and after localizing and changing variable names, we are lead to the following situation: let  $a, b, c, d \in \mathbb{Z}_{>0}$ ,  $ad > bc$  and  $\alpha, \beta \neq 0$  in an extension field  $K$  of  $k$ , and consider the univariate resultant

$$r := \text{Res}_t((x_0 + x_1 t)^a - \alpha(y_0 + y_1 t)^b, (x_0 + x_1 t)^c - \beta(y_0 + y_1 t)^d).$$

We want to show that  $x_0 y_1 - y_0 x_1$  appears with exponent  $bc$  as a factor of  $r$ .

Indeed, when  $x_1, y_1 \neq 0$ , the condition  $x_0 y_1 - y_0 x_1 = 0$  holds if and only if there exists  $t$  such that  $x_0 + x_1 t = y_0 + y_1 t = 0$ , and so  $x_0 y_1 - y_0 x_1$  occurs in  $r$  with exponent  $\mu$  equal to the intersection multiplicity at the origin of the artinian ideal  $I = \langle x^a - \alpha y^b, x^c - \beta y^d \rangle$  in  $K[x, y]$ . We claim  $\mu = bc$ .

The given equations are a Gröbner basis with leading terms  $x^a$  and  $\beta y^d$ , for the term order defined by  $\text{weight}(x) = b + d$  and  $\text{weight}(y) = a + c$ . Hence  $\dim_K K[x, y]/I = ad$ , that is, there are  $ad$  roots in the affine plane counting multiplicity. Of those,  $ad - bc$  lie in the torus, i.e., have both coordinates non-zero. No root of  $I$  has precisely one zero coordinate. Therefore the multiplicity of  $I$  at the origin is the difference  $\mu = ad - (ad - bc) = bc$ .  $\square$

**Corollary 2.2.** *The degree of a homogeneous lattice ideal  $I_B$  of codimension two can be computed from the defining  $n \times 2$ -matrix  $B$  by the following formula*

$$\text{degree}(X_B) = \beta_1 \beta_2 - \sum_{1 \leq r < s \leq n} \nu_{rs}.$$

The polynomial ring  $\mathbb{Z}[y_{i\ell}]$  has a natural  $\mathbb{Z}^n$ -grading defined by  $\deg(y_{i\ell}) = e_i$ , the  $i$ -th unit vector. The *Chow polytope*  $CP_B$  is, by definition [4, §6.3], the convex hull in  $\mathbb{R}^n$  of the degrees of all monomials appearing in the expansion of  $\tilde{\mathcal{C}}_B$ . Its faces correspond to toric deformations of the algebraic cycle  $X_B$ .

We assume that the row vectors  $b_1, b_2, \dots, b_m$  of the matrix  $B$  are ordered counterclockwise in cyclic order, and that  $b_{m+1}, \dots, b_n = 0$ . It may happen that  $b_{i+1}$  is a positive multiple of  $b_i$ . Let  $P_B$  denote the unique (up to translation) lattice polygon whose boundary consists of the directed edges  $b_1, b_2, \dots, b_m$ . For each vector  $b_i = (b_{i1}, b_{i2})$ , the linear functional

$$u = (u_1, u_2) \mapsto \det(b_i, u) = b_{i1}u_2 - b_{i2}u_1$$

attains its *minimum* value over  $P_B$  at the edge parallel to  $b_i$  for  $i = 1, \dots, m$  and is zero for  $i = m+1, \dots, n$ . Let  $\mu_i$  denote the *maximum* value of the linear functional  $u \mapsto \det(b_i, u)$  as  $u$  ranges over the polygon  $P_B$ . For  $i = 1, \dots, m$ , this maximum is attained at a unique vertex of  $P_B$  unless  $b_j = \lambda b_i$  for some  $j$  and  $\lambda < 0$ . For every lattice point  $v$  in  $P_B$ , the quantity

$$v^{(i)} := \mu_i - \det(b_i, v) \tag{2.3}$$

is a non-negative integer, invariant under translation of  $P_B$ . The vector  $(v^{(1)}, v^{(2)}, \dots, v^{(n)})$  expresses the point  $v$  in  $P_B$  in intrinsic coordinates.

**Theorem 2.3.** *The Chow polygon  $CP_B$  of a codimension 2 lattice ideal  $I_B$  is the image of the polygon  $P_B$  under the affine isomorphism  $v \mapsto (v^{(1)}, \dots, v^{(n)})$ .*

The proof of this theorem will be given in the next section, after Gale duality and duality of Plücker coordinates have been introduced. See Theorem 3.4 for the same theorem in dual formulation. Theorems 2.3 and 3.4 will then be derived from the constructions in Sections 7.1.D and 8.3.B of [4].

**Example 2.4.** For the example in the Introduction we take  $b_1 = (1, 0)$ ,  $b_2 = (1, 1)$ ,  $b_3 = (2, 2)$ ,  $b_4 = (0, 1)$ ,  $b_5 = (-1, 0)$ ,  $b_6 = (-2, 0)$ ,  $b_7 = (-1, -1)$ ,  $b_8 = (0, -1)$ ,  $b_9 = (0, -2)$  and  $P_B$  the hexagon with vertices  $(0, 0)$ ,  $(1, 0)$ ,  $(4, 3)$ ,  $(4, 4)$ ,  $(1, 4)$ ,  $(0, 3)$ . The edges of  $P_B$  are labeled by the variables as follows:  $a, \{f, i\}, b, \{d, g\}, c, \{e, h\}$ , and we have  $\mu = (4, 3, 6, 0, 0, 0, 1, 4, 8)$ . The twelve points on the boundary of  $P_B$  correspond to the twelve monomials in the expansion of  $\tilde{E}_X$ . For instance, the vertex  $v = (0, 0)$  has intrinsic coordinates  $(v^{(1)}, \dots, v^{(9)}) = (4, 3, 6, 0, 0, 0, 1, 4, 8)$  and corresponds to  $a^4 c e^4 f^3 h^8 i^6$ .

For any  $v \in P_B$ , the coordinate sum  $\sum_{i=1}^n v^{(i)}$  coincides with  $\sum_{i=1}^n \mu_i$ , and this equals the degree of the Chow form  $\tilde{C}_B$  as a polynomial in the  $y_{i\ell}$ . From this we get an alternative formula for the degree of our lattice ideal.

**Corollary 2.5.** *The degree of the variety  $X_B$  equals  $d_B = \frac{1}{2} \cdot \sum_{i=1}^n \mu_i$*

Counting lattice points in the polygon  $P_B$  gives an upper bound for the number of monomials appearing in the full discriminant  $D_X$  (see §3 below):

**Remark 2.6.** *The number of lattice points in the polygon  $P_B$  equals*

$$1 + \frac{1}{2} \left( \sum_{i=1}^n \gcd(b_{i1}, b_{i2}) + \sum_{1 \leq i < j \leq n} (b_{i2}b_{j1} - b_{i1}b_{j2}) \right)$$

*Proof.* This is a reformulation of *Pick's formula* which states that the area of a lattice polygon equals the number of lattice points in that polygon minus half the number of lattice points in its boundary, minus one.  $\square$

If the lattice ideal  $I_B$  is a complete intersection then the denominator in Theorem 2.1 is 1 and we get a determinantal formula for the Chow form, namely,  $\tilde{\mathcal{C}}_B$  equals the univariate resultant in the numerator, which can be computed as the determinant of a Sylvester or Bézoutian matrix.

It would be desirable to have a division-free determinantal formulas for the Chow form  $\tilde{\mathcal{C}}_B$  of any codimension 2 lattice ideal. At this time we know such formulas only for special classes of matrices  $B$ . We present a formula for a class which includes the example in the Introduction. Recall from [7] that the lattice ideal  $I_B$  is Cohen-Macaulay if and only if  $I_B$  is generated by the  $2 \times 2$ -minors of a  $2 \times 3$ -matrix of monomials in  $x_1, \dots, x_n$ :

$$\begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \end{pmatrix}.$$

Let  $d_i$  denote the total degree of the monomial  $m_i$ . In order for the lattice ideal  $I_B$  to be homogeneous it is necessary and sufficient that

$$d_1 + d_5 = d_2 + d_4 \quad \text{and} \quad d_1 + d_6 = d_3 + d_4.$$

For the following discussion we make an even more restrictive assumption:

$$d_1 = d_2 = d_3 \geq d_4 = d_5 = d_6. \tag{2.4}$$

We introduce four new indeterminates  $s, t, u, v$ . Let  $m_i[t]$  denote the image of the monomial  $m_i$  under the substitution  $x_i \mapsto y_{i1} + y_{i2}t$  for  $i = 1, 2, \dots, n$ . We define the *Bézout polynomial* to be the following expression:

$$\frac{1}{(s-u)(t-v)} \cdot \det \begin{pmatrix} m_1[t] + m_4[t] \cdot s & m_1[t] + m_4[t] \cdot u & m_1[v] + m_4[v] \cdot u \\ m_2[t] + m_5[t] \cdot s & m_2[t] + m_5[t] \cdot u & m_2[v] + m_5[v] \cdot u \\ m_3[t] + m_6[t] \cdot s & m_3[t] + m_6[t] \cdot u & m_3[v] + m_6[v] \cdot u \end{pmatrix}$$

Set  $\delta := d_1 + d_4$ . The Bézout polynomial can be written uniquely in the form

$$(1, v, v^2 \dots, v^{d_1-1}, u, uv, uv^2, \dots, uv^{d_4-1}) \cdot \mathbf{B} \cdot \begin{pmatrix} 1 \\ t \\ \vdots \\ t^{\delta-1} \end{pmatrix},$$

where  $\mathbf{B} = \mathbf{B}(y_{ij})$  is a certain  $\delta \times \delta$ -matrix with entries in  $k[y_{11}, y_{12}, \dots, y_{n2}]$ .

**Theorem 2.7.** *If  $I_B$  is a Cohen-Macaulay lattice ideal of codimension 2 satisfying (2.4) then its Chow form  $\tilde{C}_B$  equals the determinant of  $\mathbf{B}(y_{ij})$ .*

*Proof.* Consider the rational normal scroll of type  $(d_1, d_4)$ , a toric surface of degree  $\delta$  in a projective space of dimension  $\delta + 1$ . Its Chow form has an exact determinantal formula in terms of a Bézout matrix. A nice proof of this fact follows from recent results of Eisenbud and Schreyer [2], since the rational normal scroll is given by the  $2 \times 2$ -minors of a matrix of variables. This Chow form is the unmixed, sparse resultant for three polynomials with support

$$\{1, t, t^2, \dots, t^{d_1}, s, st, st^2, \dots, st^{d_4}\}.$$

The three polynomials  $m_i[t] + m_{i+3}[t] \cdot s$  have exactly this support. Our formula is gotten by specializing the Bézout matrix for the scroll.  $\square$

**Example 2.8.** The ideal in (1.2) satisfies the hypotheses of Theorem 2.7, with  $\delta = 4$ . The matrix (1.4) is precisely the matrix  $\mathbf{B}(y_{ij})$  in this case.  $\square$

### 3 The full discriminant

There are two different ways of presenting a toric variety of codimension two: by an  $n \times 2$ -matrix  $B$  as in [7], or by an  $(n - 2) \times n$ -matrix  $A$  as in [4, §5.1]. The two matrices are *Gale dual*, which means that the image of  $B$  equals the kernel of  $A$ . Up to this point in the paper, we have only used the  $B$ -representation. We now make a switch and introduce the  $A$ -representation.

Let  $A = (a_1, \dots, a_n)$  be an  $(n - 2) \times n$ -integer matrix of rank  $n - 2$ , and suppose there exists a vector  $w \in \mathbb{Q}^{n-2}$  such that  $w \cdot a_i = 1$  for  $i = 1, 2, \dots, n$ . We can choose an integral  $n \times 2$  matrix  $B$  whose columns are a  $\mathbb{Z}$ -basis of  $\ker_{\mathbb{Z}}(A)$ . The matrix  $B$  has rank 2 and  $A \cdot B = 0$ . It is unique modulo right



multiplication by  $GL(2, \mathbb{Z})$ . Let  $I_A = I_B$  denote the corresponding toric ideal in  $k[x_1, \dots, x_n]$  and  $X = X_A = X_B$  the corresponding toric variety in  $\mathbf{P}^{n-1}$ .

Here it is important to note that not all integer matrices  $B$  arise as the Gale dual of some matrix  $A$  as above. For this it is necessary and sufficient that  $\mathbb{Z}^n / \text{im}_{\mathbb{Z}}(B)$  is torsion-free, or equivalently, that the ideal  $I_B$  is prime.

The  $A$ -discriminant  $D_A$  is an irreducible polynomial in  $\mathbb{Z}[x_1, \dots, x_n]$  which vanishes under a specialization if the corresponding Laurent polynomial

$$f = \sum_{i=1}^n x_i \cdot t_1^{a_{i1}} t_2^{a_{i2}} \cdots t_{n-2}^{a_{i,n-2}} \quad \text{where } x_1, \dots, x_n \in \mathbb{C}^*$$

has a multiple root  $(t_1, \dots, t_{n-2})$  in  $(\mathbb{C}^*)^{n-2}$ . Equivalently, the hypersurface  $\{D_A = 0\}$  is projectively dual to the toric variety  $X$ , when the dual variety  $X^\vee$  is a hypersurface, and  $D_A = 1$  otherwise; see [4, §1.1 and §9.1].

In the next section we give a formula for the  $A$ -discriminant  $D_A$  and its degree. In this section, we study a larger polynomial  $E_A$  which contains  $D_A$  as a factor. It is called the *principal  $A$ -determinant* in [4] but we prefer the term *full discriminant*. Actually, our full discriminant agrees with expression (1.1) in [4, 10.1.A], but there is a slight inaccuracy in [4, Theorem 10.1.2] since  $E_A$  does not generally have content 1. An extra integer factor is needed. This integer factor would be  $2^{14}$  for the example (1.7) in the Introduction.

Before stating the definition of  $E_A$ , we first review the duality between primal and dual Plücker coordinates, and see how it ties in with Gale duality. For  $1 \leq i < j \leq n$ , let  $B(i, j)$  the submatrix of  $B$  consisting of the  $i$ -th and  $j$ -th rows, and let  $A\langle i, j \rangle$  denote the submatrix of  $A$  gotten by omitting the  $i$ -th and  $j$ -th columns. Here signs are adjusted so that  $\det A\langle i, j \rangle = \det B(i, j)$ , up to a global constant. In Section 2 we used an  $n \times 2$  matrix  $Y = (y_{il})$  of indeterminates. The *dual Plücker coordinates* of a line in  $\mathbf{P}^{n-1}$  are

$$[i j] := \det Y(i, j) = y_{i1}y_{j2} - y_{i2}y_{j1} \quad \text{for } 1 \leq i < j \leq n. \quad (3.1)$$

Here we consider an  $(n-2) \times n$ -matrix  $Z = (z_{ij})$  of indeterminates. The *primal Plücker coordinates* of our line are the  $(n-2) \times (n-2)$ -subdeterminants

$$\langle i j \rangle = \det Z\langle i, j \rangle \quad (\text{with the sign adjusted as usual}).$$

The dual Chow form  $\tilde{\mathcal{C}}_B$  is a polynomial of degree  $d_B$  in the brackets (3.1). Replacing  $[i j] \mapsto \langle i j \rangle$  in  $\tilde{\mathcal{C}}_B$  gives a homogeneous polynomial of degree  $(n-2)d_B$  in the variables  $z_{ij}$ . It is denoted  $\mathcal{C}_A$  and called the *primal Chow form*. Note that  $\mathcal{C}_A$  coincides with the  $A$ -resultant defined in [4, §8.2.A].

**Definition 3.1.** The *full discriminant*  $E_A$  is the image of the primal Chow form  $\mathcal{C}_A$  under the specialization  $z_{ij} \mapsto a_{ij}x_j$  for  $i = 1, \dots, n-2$ ,  $j = 1, \dots, n$ .

We next show how to compute the full discriminant directly from the dual Chow form  $\tilde{\mathcal{C}}_B$  and hence from the formulas in Theorems 2.1 and 2.7.

**Proposition 3.2.** *The full discriminant  $E_A$  and the dual Chow form  $\tilde{\mathcal{C}}_B(y_{i\ell})$  are related by the following formula:*

$$E_A(x_1, \dots, x_n) = (x_1 \dots x_n)^{d_B} \cdot \tilde{\mathcal{C}}_B(b_{i\ell}/x_i, i = 1, \dots, n, \ell = 1, 2). \quad (3.2)$$

The exponent  $d_B$  is the degree of the toric variety  $X$  and hence coincides with the normalized volume of the  $(n-3)$ -dimensional polytope  $\text{conv}(A)$ . Gale dual formulas for this volume are given in Corollaries 2.2 and 2.5.

*Proof.* The specialization  $z_{ij} \mapsto a_{ij}x_j$  in Definition 3.1 is equivalent to

$$\langle r \ s \rangle \rightarrow \det A \langle r, s \rangle \prod_{k \neq r, s} x_k \quad \text{for } 1 \leq r < s \leq n \quad (3.3)$$

at the level of primal Plücker coordinates. The dual Chow form  $\tilde{\mathcal{C}}_B$  is a  $\mathbb{Z}$ -linear combination of bracket terms  $\prod [r \ s]$  of degree  $d_B$ . If we substitute  $b_{i\ell}/x_i$  for  $y_{i\ell}$  in the expansion of such a bracket term  $\prod [r \ s]$  then we get

$$\begin{aligned} \prod [r \ s] &\longrightarrow \prod (\det B(r, s)/(x_r x_s)) = \prod (\det A \langle r, s \rangle / (x_r x_s)) = \\ &= (x_1 \dots x_n)^{-d_B} \cdot \prod (\det A \langle r, s \rangle \prod_{k \neq r, s} x_k) \longleftarrow (x_1 \dots x_n)^{-d_B} \cdot \prod \langle r \ s \rangle \end{aligned}$$

Hence the specialized dual Chow form on right hand side of (3.2) equals the specialization of the primal Chow form  $\mathcal{C}_A$  under (3.3), as desired.  $\square$

It is known from [4, Theorem 10.1.2] that the full discriminant  $E_A$  is a product of irreducible factors  $D_{A'}$  where  $A'$  ranges over facial discriminants. In particular, each monomial  $x_i$  corresponding to a vertex  $a_i$  of  $\text{conv}(A)$  appears to some positive power in the factorization of  $E_A$ . It is curious to note that the monomial factors disappear when we pass to dual coordinates. We define the *dual full discriminant* by specializing the dual Chow form:

$$\tilde{E}_B(x_1, \dots, x_n) = \tilde{\mathcal{C}}_B(b_{i\ell} \cdot x_i, i = 1, \dots, n, \ell = 1, 2). \quad (3.4)$$

Proposition 3.2 is equivalent to the reciprocity formula:

$$\tilde{E}_B(x_1, \dots, x_n) = (x_1 \dots x_n)^{d_B} \cdot E_A(1/x_1, \dots, 1/x_n). \quad (3.5)$$

**Lemma 3.3.** *The dual full discriminant  $\tilde{E}_B$  has no monomial factors.*

*Proof.* Suppose that the variable  $x_i$  divides  $\tilde{E}_B$ . Then every bracket monomial appearing in the dual Chow form  $\tilde{C}_B$  contains the letter  $i$ . Equivalently, every bracket monomial in the primal Chow form  $C_A$  contains a bracket  $\langle r s \rangle$  with  $r = i$  or  $s = i$ . In view of [4, Theorem 8.3.3], this means that every regular triangulation of  $A$  contains a simplex for which  $a_i$  is not a vertex. But this is false, since  $a_i$  lies in every maximal simplex of the *reverse lexicographic triangulation* of  $A$ , for  $x_i$  smallest; see [10, Proposition 8.6].  $\square$

The *secondary polygon*  $\Sigma(A)$  of the configuration  $A$  coincides with the Newton polygon of the full discriminant  $E_A$ , by [4, Theorem 10.1.4]. It is a 2-dimensional convex polytope lying in  $\mathbb{R}^n$ . Let  $P_B$  be the polygon considered in Section 2. For  $v \in P_B$ , let  $(v^{(1)}, \dots, v^{(n)})$  be the vector defined in (2.3).

**Theorem 3.4.** *The secondary polytope  $\Sigma(A)$  is the image of the polygon  $P_B$  under the affine isomorphism which sends  $v$  to  $(d_B - v^{(1)}, \dots, d_B - v^{(n)})$ .*

*Proof.* It suffices to prove this theorem for the case when all  $b_i$  are non-zero. Indeed, if  $b_{m+1} = \dots = b_n = 0$  then [4, Theorem 10.1.2] implies that

$$E_A(x_1, \dots, x_n) = (x_{m+1} \dots x_n)^{d_B} \cdot E_{A'}(x_1, \dots, x_m),$$

where  $A'$  is a Gale dual of the configuration  $(b_1, \dots, b_m)$ . Our assertion for  $\Sigma(A')$  implies that for  $\Sigma(A)$ . We hence assume that  $b_i \neq 0$  for all  $i = 1, \dots, n$ .

Each vertex  $w = (w_1, \dots, w_n)$  of  $\Sigma(A)$  corresponds uniquely to the Gale dual of a regular triangulation  $\Delta_w$ , and hence to a pair of adjacent linearly independent vectors  $b_k, b_{k+1}$  (indices are understood modulo  $n$ ; recall that  $\sum_i b_i = 0$ ). By [4, Definition 7.1.6], the  $i$ -th coordinate of  $w$  equals the sum of the normalized volumes of those simplices in  $\Delta_w$  which contain the point  $a_i$ . By Gale duality,  $w_i = \sum_{r,s} |\det(b_r, b_s)|$  where the sum is over all indices  $r \neq i, s \neq i$  such that  $b_k$  and  $b_{k+1}$  lie in the cone spanned by  $b_r$  and  $b_s$ . Let  $v_w$  be the vertex of  $P_B$  between the edges parallel to  $b_k$  and  $b_{k+1}$ . We claim that  $v_w \in \mathbb{Z}^2$  is mapped to  $w \in \mathbb{Z}^n$  under the affine isomorphism given above.

We note that the maximum  $\mu_i$  of the values  $\det(b_i, v)$  is attained at the vertex  $v \in P_B$  between the edges parallel to two independent vectors  $b_\ell, b_{\ell+1}$  such that  $\det(b_i, b_\ell) \geq 0$  and  $\det(b_i, b_{\ell+1}) < 0$  (indices modulo  $n$ ). What we are claiming is the identity  $w_i = d_B - \det(b_i, v) + \det(b_i, v_w)$ .

Let  $C_k$  denote the set of index pairs  $(r, s)$  such that  $b_k$  and  $b_{k+1}$  lie in the cone spanned by  $b_r$  and  $b_s$ . The set  $C_k$  is Gale dual to our regular

triangulation, and, hence  $d_B = \text{vol}(\text{conv}(A))$  equals  $\sum_{(r,s) \in C_k} |\det(b_r, b_s)|$ . If we start drawing  $P_B$  from the origin, then,  $v = \sum_{j=1}^{\ell} b_j$  and  $v_w = \sum_{j=1}^k b_j$ . Our assertion takes the following form:

$$\sum_{(r,s) \in C_k, r \neq i, s \neq i} |\det(b_r, b_s)| = \sum_{(r,s) \in C_k} |\det(b_r, b_s)| - \sum_{j=1}^{\ell} \det(b_i, b_j) + \sum_{j=1}^k \det(b_i, b_j).$$

After erasing equal terms on both sides, the following remains to be proved:

$$\sum_{j=1}^{\ell} \det(b_i, b_j) - \sum_{j=1}^k \det(b_i, b_j) = \sum_{j: (i,j) \in C_k} |\det(b_i, b_j)|.$$

The proof is straightforward by a case distinction involving the relative positions of the vectors  $b_i, b_k, b_{k+1}$  and  $b_{\ell}$  in the plane.  $\square$

*Proof of Theorem 2.3:* If  $b_1, \dots, b_n$  span the lattice  $\mathbb{Z}^2$  then we find a corresponding matrix  $A$ , and Theorem 2.3 follows directly from Theorem 3.4 and the reciprocity formula (3.5). Otherwise, the scheme  $X_B$  is the equidimensional union of  $r > 1$  torus translates of a fixed toric variety  $X_{B'}$ . Following [4, §4.1.A], the Chow form  $\tilde{C}_B$  factors into  $r$  irreducible polynomials, each of which is a torus translate of the irreducible Chow form  $\tilde{C}_{B'}$ . Therefore the Chow polygon  $CP_B$  equals  $r \cdot CP_{B'}$ . The configuration  $B'$  is  $GL(\mathbb{R}^2)$ -equivalent to  $B$ , and it does possess a Gale dual  $A'$ . Our assertion holds for  $CP_{B'}$  and it follows for  $CP_B$  by scaling.  $\square$

Let us now take a look at what happens to the formula in Theorem 2.1 under the specialization  $y_{i\ell} \mapsto b_{i\ell} \cdot x_i$  in (3.4). A line through the origin in  $\mathbb{R}^2$  is said to be *relevant* if it contains two vectors  $b_r, b_s$  in opposite directions. So, if the rows of  $B$  are in general position, then there are no relevant lines. The example in the introduction has three relevant lines.

Consider the specializations of the two polynomials  $H_{\ell}(t)$  in (2.2):

$$h_{\ell}(t) := \prod_{i: b_{i\ell} > 0} (b_{i1} + b_{i2}t)^{b_{i\ell}} x_i^{b_{i\ell}} - \prod_{i: b_{i\ell} < 0} (b_{i1} + b_{i2}t)^{-b_{i\ell}} x_i^{-b_{i\ell}}, \quad \ell = 1, 2. \quad (3.6)$$

**Remark 3.5.** *The polynomials  $h_1, h_2$  have a common factor if and only if there exist a relevant line which is not a coordinate axis.*

The presence of two vectors  $b_r, b_s$  in opposite directions in the interior of two quadrants then causes the resultant  $\text{Res}_t(h_1, h_2)$  to vanish. Also,  $\det(B(r, s)) = 0$ , while  $\nu_{rs} \neq 0$ . When there are two opposite vectors on a coordinate axis, both numbers are zero and  $\det(B(r, s))^{\nu_{rs}} = 1$ . We deduce:

**Proposition 3.6.** *Assume there are no relevant lines for the configuration  $B$  except for the coordinate axes. Then the dual full discriminant equals*

$$\tilde{E}_B = \frac{\text{Res}_t(h_1, h_2)}{\prod_{1 \leq r < s \leq n} \det(B(r, s))^{\nu_{rs}} \prod_{1 \leq r < s \leq n} (x_r \cdot x_s)^{\nu_{rs}}}.$$

In the next section we will show how to use Theorem 2.1 to compute discriminants even if the hypothesis of the above proposition is not satisfied.

## 4 The $A$ -discriminant

Let  $A \in \mathbb{Z}^{(n-2) \times n}$  and  $B \in \mathbb{Z}^{n \times 2}$  be Gale dual matrices as before, and let  $X$  be the corresponding toric variety of codimension 2 in  $\mathbf{P}^{n-1}$ . The  $A$ -discriminant  $D_A$  is the defining irreducible polynomial of the dual variety  $X^\vee$ , unless  $\text{codim}(X^\vee) > 1$  in which case  $D_A = 1$ . Gel'fand, Kapranov and Zelevinsky [4, Theorem 10.1.2] proved that  $D_A$  appears with exponent one in the factorization of the full discriminant  $E_A$ . In this section we compute  $D_A$  and all other factors of  $E_A$  in terms of the row vectors  $b_i \in \mathbb{R}^2$  of  $B$ .

Throughout this section we shall assume that  $b_i \neq 0$  for  $i = 1, \dots, n$ . This means that  $X$  is not a cone over a coordinate point, or that  $X^\vee$  does not lie in a coordinate hyperplane. All results in Section 4 require this hypothesis.

Each relevant line in the plane is identified with one of the two primitive vectors  $v \in \mathbb{Z}^2$  on that line. We abbreviate  $b_i^{(v)} := \det(b_i, v)$ . With each such line  $v$ , we associate a codimension one discriminant as in (1.1).

$$D_v := \prod_{j: b_j^{(v)} < 0} (b_j^{(v)})^{-b_j^{(v)}} \prod_{i: b_i^{(v)} > 0} x_i^{b_i^{(v)}} - \prod_{i: b_i^{(v)} > 0} (b_i^{(v)})^{b_i^{(v)}} \prod_{j: b_j^{(v)} < 0} x_j^{-b_j^{(v)}} \quad (4.1)$$

Let  $b_{i_1}, \dots, b_{i_s}$  be all the row vectors of  $B$  which lie on the relevant line  $v$ . There is a unique integer vector  $(\lambda_1, \dots, \lambda_s)$  such that  $b_{i_j} = \lambda_j \cdot v$  for  $j = 1, \dots, s$ . We direct the primitive vector  $v \in \mathbb{Z}^2$  so that the coordinate sum  $\alpha_v := \lambda_1 + \dots + \lambda_s$  is nonnegative, and we define  $\delta_v := \sum \{-\lambda_i : \lambda_i < 0\}$ . Using this notation, Remark 3.5 can now be refined as follows:

**Remark 4.1.** If  $v = (v_1, v_2)$  is a relevant line for  $B$  then  $v_1 + v_2 t$  appears with exponent  $\delta_v \cdot v_i$  in the factorization of the polynomial  $h_i(t)$  in (3.6).

Denote by  $p_1(t), p_2(t)$  the respective remaining factors, that is,

$$h_\ell(t) = p_\ell(t) \cdot \prod_{v \text{ relevant}} (v_1 + v_2 t)^{\delta_v \cdot v_\ell} \quad \ell = 1, 2. \quad (4.2)$$

Now the resultant  $r_B := \text{Res}_t(p_1, p_2)$  is a non-zero polynomial in  $x_1, \dots, x_n$ . It is customary to call  $r_B$  the *residual resultant* of  $h_1$  and  $h_2$ . We shall prove the following formulas for the full discriminant and the  $A$ -discriminant.

**Theorem 4.2.** *There exist monomials  $x^u, x^{u'}$  and integers  $\nu, \nu'$  such that*

$$\begin{aligned} D_A(x_1, \dots, x_n) &= (1/\nu) \cdot x^u \cdot r_B(1/x_1, \dots, 1/x_n) & \text{and} \\ E_A(x_1, \dots, x_n) &= \nu' \cdot x^{u'} \cdot D_A(x_1, \dots, x_n) \cdot \prod_{v \text{ relevant}} D_v(x_1, \dots, x_n)^{\delta_v} \end{aligned}$$

*Proof.* We shall first prove the following claim about the full discriminant:

$$r_B(1/x_1, \dots, 1/x_n) \cdot \prod_{v \text{ relevant}} D_v(x_1, \dots, x_n)^{\delta_v} \text{ divides } E_A(x_1, \dots, x_n)$$

in the Laurent polynomial ring  $k[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$ .

Fix any relevant line  $v$ . Choose an isomorphism in  $SL_2(\mathbb{Z})$  which maps  $v$  to  $(0, 1)$ , and apply this isomorphism to the rows of  $B$ . Also reorder the rows of  $B$  so that the multiples of  $v$  come first. After this transformation, the first column of  $B$  has the entries  $0, \dots, 0, b_{s+1}^{(v)}, b_{s+2}^{(v)}, \dots, b_n^{(v)}$ .

For  $\ell = 1, 2$  and  $i = 1, \dots, s$  **only**, substitute  $y_{i\ell} = b_{i\ell}/x_i$  into the Chow form  $\mathcal{C}_B$ . Let  $\tilde{H}_\ell$  be the polynomials resulting from  $H_\ell$  in (2.2) under the same substitution. Then  $\tilde{H}_1 = H_1$ , but  $\tilde{H}_2$  is divisible by  $t^{\delta_v}$ , and this is the highest possible power of  $t$  with this property (cf. Remark 4.1). Theorem 2.1 implies that the specialized Chow form factors, and one of its factors is

$$\text{Res}_t(H_1, t^{\delta_v}) = (H_1(0))^{\delta_v} \quad (4.3)$$

For all subsequent specializations, the Chow form factors accordingly. When we substitute  $y_{i\ell} = b_{i\ell}/x_i$  for  $i = s+1, \dots, n$ ,  $\ell = 1, 2$ , into  $H_1(0)$  then we get the binomial  $D_v$  in (4.1). Clearly, the residual resultant  $r_B$  divides the full discriminant  $\tilde{E}_B$ . The above claim follows from this. Moreover, our argument shows that  $D_v^{\delta_v}$  is the highest power of  $D_v$  which divides  $E_A$ .

Consider now the factorization formula given by Gel'fand, Kapranov and Zelevinsky in [4, Theorem 10.1.2]. Under Gale duality, the proper faces of the polytope  $\text{conv}(A)$  which are not simplices correspond to relevant lines  $v$ , and their face discriminants are precisely the binomials  $D_v$ . In other words, the full discriminant  $E_A$  equals the  $A$ -discriminant  $D_A$  times the product of the expressions  $D_v^{\delta_v}$  where  $v$  ranges over all relevant lines. We conclude from our claim that  $r_B(1/x_1, \dots, 1/x_n)$  divides  $D_A(x_1, \dots, x_n)$  in the Laurent polynomial ring. Since  $D_A$  is irreducible, both of our assertions follow.  $\square$

We next compute the Newton polygon of the  $A$ -discriminant. Define

$$b_v := \alpha_v \cdot v = b_{i_1} + \dots + b_{i_s}$$

for any relevant line  $v$ . It may happen that  $b_v = 0$ . We take all-non zero vectors  $b_v$  and all vectors  $b_i$  which do not lie in relevant lines, and we order them counterclockwise in cyclic order. Let  $Q_B$  denote the unique (up to translation) lattice polygon whose boundary consists of these directed edges. For any  $i = 1, \dots, n$  and any lattice point  $v$  in  $Q_B$ , we define

$$\nu_i := \min\{\det(b_i, u), u \in Q_B\} \quad \text{and} \quad \bar{v}^{(i)} := \det(b_i, v) - \nu_i. \quad (4.4)$$

Hence,  $\bar{v}^{(i)} \in \mathbb{Z}_{\geq 0}$  is the normalized lattice distance from  $v$  to the boundary of  $Q_B$ , in the direction orthogonal to  $b_i$ .

**Theorem 4.3.** *The Newton polygon  $N(D_A)$  of the  $A$ -discriminant  $D_A$  is the image of the polygon  $Q_B$  under the affine isomorphism  $v \mapsto (\bar{v}^{(1)}, \dots, \bar{v}^{(n)})$ .*

*Proof.* Suppose first that there are no relevant lines. Then,  $Q_B = P_B$ , and the secondary polygon  $\Sigma(A)$  and the Newton polygon  $N(D_A)$  are equal up to translation. More precisely,  $\Sigma(A) = N(D_A) + \alpha$  where  $\alpha_i$  is the exponent of  $x_i$  as a factor of  $E_A$ . Using [4, Theorem 10.1.2] and Gale duality, we find

$$\alpha_i = d_B - \sum_{j: \det(b_i, b_j) > 0} \det(b_i, b_j).$$

In light of Theorem 3.4, it suffices to show that

$$d_B - \mu_i + \det(b_i, v) = \alpha_i + \det(b_i, v) - \nu_i \quad \text{for all } v \in Q_B, i = 1, \dots, n.$$

After cancelling terms common to both sides, what remains to be shown is

$$\sum_{j: \det(b_i, b_j) > 0} \det(b_i, b_j) = \mu_i - \nu_i.$$

This identity holds because both sides are equal to the normalized lattice width of the polygon  $Q_B = P_B$  in the direction orthogonal to  $b_i$ .

We next assume that relevant lines exist. Then  $\nu_i$  generally differs from  $\nu'_i := \min\{\det(b_i, u), u \in P_B\}$ . The secondary polytope  $\Sigma(A)$  equals  $N(D_A) + \alpha$  plus the Minkowski sum of the Newton segments of the binomials (4.1) where  $v$  runs over all relevant lines. Hence, up to lattice translation,

$$P_B = Q_B + \sum_{v \text{ relevant}} \text{conv}\{0, v\}. \quad (4.5)$$

The minimum value of the linear functional  $\det(b_i, *)$  over the line segment  $\text{conv}\{0, v\}$  is  $\det(b_i, v)$ , when this value is negative and zero otherwise. Therefore (4.5) translates into the identity

$$\nu'_i = \nu_i + \sum_{v \text{ relevant}} \delta_v \cdot \min\{0, \det(b_i, v)\} \quad \text{for } i = 1, \dots, n.$$

The argument for the case of no relevant lines now completes the proof.  $\square$

We deduce the following formula for the degree of the  $A$ -discriminant:

**Corollary 4.4.**

$$\text{degree}(D_A) = - \sum_{i=1}^n \nu_i$$

We can also extract the following characterization from Theorem 4.3.

**Corollary 4.5.** *The  $A$ -discriminant  $D_A$  is equal to 1 if and only if the polygon  $P_B$  is centrally symmetric.*

*Proof.* The condition  $D_A = 1$  is equivalent to  $Q_B$  being a point. This happens if and only if all vectors  $b_i$  lie in a relevant line, and  $\alpha_v = 0$  for each relevant line  $v$ . This last condition is equivalent to  $P_B$  being centrally symmetric.  $\square$

The following variant to the formula of Theorem 4.2 works well in practice for computing the  $A$ -discriminant  $D_A$ . In the affine plane with coordinates  $(w_1, w_2)$ , consider the following parametrically presented rational curve:

$$w_\ell = \prod_{i=1}^n (b_{i1} + b_{i2}t)^{b_{i\ell}}, \quad \ell = 1, 2. \quad (4.6)$$



This is the *Horn uniformization* in [4, §9.3.C]. Let  $\Delta(w_1, w_2)$  be the irreducible polynomial defining this curve. This is a dehomogenization of the  $A$ -discriminant, by Theorem 4.2 or by [4, Theorem 9.3.3. (a)]. More precisely,

$$D_A(x_1, \dots, x_n) = (\text{a monomial}) \cdot \Delta\left(\prod_{i=1}^n x_i^{b_{i1}}, \prod_{i=1}^n x_i^{b_{i2}}\right). \quad (4.7)$$

The common factors in the numerator and denominator of (4.6) are precisely the relevant lines which are not a coordinate axis. In other words, cancelling common factors in (4.6) is equivalent to replacing  $h_i(t)$  by  $p_i(t)$  in (4.2).

We can get a description of the Newton polygon  $N(\Delta)$  of  $\Delta(w_1, w_2)$  by “dehomogenizing” the result in Theorem 4.3 as follows. Let  $b_i^\perp := (b_{i2}, -b_{i1})$  and note that  $-\det(b_i, v)$  equals the inner product  $\langle b_i^\perp, v \rangle$ .

**Corollary 4.6.** *Let  $B^\perp = \{b_1^\perp, \dots, b_n^\perp\}$  and consider the polygon  $Q_{B^\perp}$  translated so that it lies in the first quadrant and its boundary intersects both coordinate axes. Then  $N(\Delta) = Q_{B^\perp}$ .*

This result has been obtained independently by Sadykov [8], under the hypothesis that there are no relevant lines outside the coordinate axes.

**Example 4.7.** We consider the toric 3-fold of degree 43 in  $\mathbf{P}^5$  which appears as Example 5.10 in [7]. It is defined by the  $6 \times 2$  integer matrix  $B$  with row vectors  $(2, 3), (-1, 4), (-5, 1), (3, -1), (2, -3), (3, -2)$ . The lattice ideal  $I_B$  has seven minimal generators. There are no relevant lines. The polygon  $P_B = Q_B$  is a hexagon. Using Remark 2.6 we find that  $Q_B$  contains 40 lattice points. They correspond to the 40 terms in the  $A$ -discriminant  $D_A$ . The 6 vertices of  $P_B$  correspond to the various leading terms in  $D_A$ . Using (4.7) in any computer algebra system we easily compute:

$$\begin{aligned} D_A = & - (7)^7 (17)^{17} (19)^{19} x_1^{16} x_4^{11} x_5^{23} x_6^{22} \\ & - (2)^{34} (3)^{15} (5)^{15} (13)^{13} x_1^{20} x_2^{36} x_3^{11} x_6^5 \\ & + (2)^{10} (5)^{15} (11)^{11} (17)^{17} x_1^{23} x_2^{19} x_5^{13} x_6^{17} \\ & + (2)^{64} (7)^{14} (13)^{13} x_3^{19} x_4^{28} x_5^{16} x_6^9 \\ & + (3)^{21} (7)^7 (11)^{11} (13)^{13} x_2^{16} x_3^{26} x_4^{25} x_5^5 \\ & - (2)^{10} (5)^{15} (11)^{11} (17)^{17} x_1^9 x_2^{29} x_3^{21} x_4^{13} \\ & + \text{interior terms.} \end{aligned}$$

We invite the reader to draw  $Q_B$  and verify Theorem 4.3 for this example.  $\square$

## 5 Resultants having Newton triangles

Mixed resultants form a subclass among all discriminants, by the Cayley trick of elimination theory [4, §9.1.A]. This subclass is important for the theory of hypergeometric functions: conjecturally, it consists of the denominators of rational hypergeometric functions [1, Conjecture 1.4]. In this section we examine the Cayley construction and mixed resultants in codimension 2.

The Gale dual of a Cayley configuration  $A$  is a  $(2r + 3) \times 2$ -matrix

$$B = (b_1, b_2, \dots, b_r, c_1, c_2, -b_1, \dots, -b_r, -c_1 - c_2)^T,$$

where the rows of the submatrix  $\tilde{B} := (b_1, b_2, \dots, b_r, c_1, c_2)^T$  span  $\mathbb{Z}^2$ . We assume that all  $b_i$  are non-zero and  $\det(c_1, c_2) \neq 0$ . By Corollary 4.5,  $D_A \neq 1$ .

Fix an  $r \times (r + 2)$ -matrix Gale dual to  $\tilde{B}$  whose left  $r \times r$ -minor is diagonal:

$$\tilde{A} = \begin{pmatrix} \gamma_1 & & \alpha_1 & \beta_1 \\ & \ddots & \vdots & \vdots \\ & & \gamma_r & \alpha_r & \beta_r \end{pmatrix} \quad \text{where } \gamma_i \in \mathbb{Z}_{>0} \text{ and } (\alpha_i, \beta_i) \in \mathbb{Z}^2 \setminus \{(0, 0)\}.$$

This matrix lifts to a  $(2r + 1) \times (2r + 3)$ -matrix Gale dual to  $B$  as follows:

$$A = \begin{pmatrix} \tilde{A} & 0 \\ I_{r+1}e_{r+1} & I_{r+1} \end{pmatrix}, \quad (5.1)$$

where  $I_{r+1}$  is the unit matrix of size  $r + 1$  and  $e_{r+1} = (0, 0, \dots, 0, 1)^T$ . The columns of  $A$  index the coefficients in a sparse system of  $r + 1$  equations:

$$\begin{aligned} f_0 &= z_1 \cdot t_1^{\alpha_1} \cdots t_r^{\alpha_r} + z_2 \cdot t_1^{\beta_1} \cdots t_r^{\beta_r} + z_3 \\ f_i &= x_i \cdot t_i^{\gamma_i} + y_i \quad \text{for } i = 1, \dots, r. \end{aligned}$$

This system consists of  $r$  binomials and one Laurent trinomial, as in (1.8). The sparse resultant  $\text{Res}(f_0, f_1, \dots, f_r)$  is the unique (up to sign) irreducible polynomial in  $x_1, \dots, x_r, y_1, \dots, y_r, z_1, z_2, z_3$  which vanishes when the system has a common root  $(t_1, \dots, t_r)$  in the  $r$ -torus. From [4, Prop. 9.1.7] we get:

**Remark 5.1.** *The  $A$ -discriminant  $D_A$  equals the sparse resultant of  $f_0, \dots, f_r$ .*

We now apply the product formula for resultants [6], which amounts to evaluating  $f_0$  at the common zeros of  $f_1, \dots, f_r$ . The number of zeroes equals

$$\Gamma := \gamma_1 \gamma_2 \cdots \gamma_r = |\det(c_1, c_2)|.$$

Let  $\eta_i$  denote a primitive  $\gamma_i$ -th root of unity. The product formula implies:

**Proposition 5.2.** *Up to a Laurent monomial factor, the  $A$ -discriminant is*

$$D_A = \text{monomial} \cdot \prod_{i_1=1}^{\gamma_1} \cdots \prod_{i_r=1}^{\gamma_r} f_0(\eta_1^{i_1} z_1, \eta_2^{i_2} z_2, \dots, \eta_r^{i_r} z_r)$$

where  $z_i = \left(-\frac{y_i}{x_i}\right)^{1/\gamma_i}$  for  $i = 1, \dots, r$ .

Since  $f_0$  is a trinomial, this formula gives an upper bound of  $\binom{\Gamma+2}{2}$  for the number of terms appearing in the expansion of  $D_A = \text{Res}(f_0, \dots, f_r)$ . This bound is quadratic in  $\Gamma$ . In truth, this number grows linearly in  $\Gamma$ .

**Theorem 5.3.** *The number of terms appearing in  $D_A$  is at most  $\frac{5}{4} \cdot \Gamma + \frac{7}{4}$ .*

This bound is tight if the vectors  $c_1$  and  $c_2$  span the lattice  $\mathbb{Z}^2$ . In this case,  $\Gamma = \det(c_1, c_2) = 1$  and the resultant  $D_A$  has three terms. It is also tight for the example in the Introduction, where  $\Gamma = 4$  and  $D_A$  has six terms.

*Proof.* According to Theorem 4.3, the Newton polygon of the discriminant  $D_A$  is essentially the lattice triangle  $Q_B = \text{conv}\{0, c_1, c_2\}$ . The number of terms in  $D_A$  is at most the number of lattice points in  $Q_B$ . Using Pick's formula as in Remark 2.6, we find that the number  $\#(Q_B \cap \mathbb{Z}^2)$  equals

$$1 + \frac{1}{2} \cdot (|\det(c_1, c_2)| + \gcd(c_{11}, c_{12}) + \gcd(c_{21}, c_{22}) + \gcd(c_{11} + c_{21}, c_{12} + c_{22})).$$

Using the inequality  $a + b \leq ab + 1$ , we find that the sum of any two of the three last summands is bounded above by  $\Gamma + 1 = |\det(c_1, c_2)| + 1$ . Therefore,

$$\#(Q_B \cap \mathbb{Z}^2) \leq 1 + \frac{1}{2} \cdot \left(\Gamma + \frac{3}{2} \cdot (\Gamma + 1)\right).$$

This is the desired inequality. □

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